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INFLUENCE FUNCTIONS FOR STRESS AND DISPLACEMENT DISCONTINUITY ELEMENTS IN AN ANISOTROPIC MEDIUM

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SUMMARY

Influence functions, that permit us to determine stresses and displacements at an arbitrary point in an infinite, homogeneous, linear elastic, anisotropic medium due to different three-dimensional (3-D) stress or displacement discontinuities distributed on infinite, flat, band-type elements, are presented. Any straight-line segment on the band, which is perpendicular to its infinite side, has the same distribution of the discontinuities. Along with the functions, their Taylor series approximations are also provided. The last can be useful to analyse stresses and displacements at points distant from the elements. The functions allow us to avoid procedures of numerical integration in the Indirect Boundary Element Method and/or the Displacement Discontinuity Method computer codes that are able to solve complete plane-strain problems with 3-D boundary conditions for an elastic, anisotropic medium. © 1997 by John Wiley & Sons, Ltd.

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Key words: anisotropic elastic medium; complete plane-strain problem; influence function; stress discontinuity element; displacement discontinuity element

1. INTRODUCTION

Various numerical methods are widely used in Geomechanics. The Indirect Boundary Element Method (IBEM) and the Displacement Discontinuity Method (DDM) are two of them.

One of the main steps for creating the IBEM or the DDM computer codes is the integration of force or dipole actions on boundary elements. A numerical integration results in approximate values of stress and displacement components. Error magnitudes are unknown and they can be extremely high in close vicinity of the boundaries because the solution for a concentrated force or a dipole has a singularity at the point of application. It is possible to eliminate this kind of error by employing either analytical influence functions for boundary elements or methods that allow us to avoid such a numerical integration in the codes.

Analytical influence functions for 2-D constant stress discontinuity elements in an isotropic medium were found by Veryuzhskii[1] and Antonov and Zinov'ev.[2] Crouch[3] derived functions for

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2-D constant displacement discontinuity elements in an isotropic medium. Two-dimensional higher order and crack tip elements in an isotropic medium were presented by Crawford and Curran[4] and Chan.[5] Brady and Bray[6,7] found analytical influence functions for stress and displacement discontinuity elements that can be used to solve complete plane strain problems with 3-D boundary conditions in case of an isotropic elastic medium. Analytical influence functions for 2-D constant stress discontinuity elements in an orthotropic medium were presented by Aitaliev and Kayupov,[8] Crouch and Starfield,[9] and Aitaliev *et al.*[10] These functions, being derived in closed form, produce distinct advantages over most other approaches employing numerical integration. Some of these advantages are noted by Chan in paper:[5] (1) the analytical influence functions are exact while unknown errors occur during numerical integration, (2) storage requirement is lowered and computation speed is raised, (3) analytical influence functions can be examined exactly.

Unfortunately, if a medium with arbitrary anisotropy is considered, our ability to derive similar analytical expressions is limited because closed-form Kelvin's solution for such a medium has not yet been found. Nevertheless, even in this case, we are able to carry out analytical integration of relevant expressions and get functions that describe integrated action of forces or dipoles on boundary elements. These functions are presented herein. Being employed in the IBEM and/or the DDM computer codes, they allow us to avoid procedures of numerical integration in the codes and, as a result, diminish computational errors, especially, close to boundaries. The codes will be able to solve complete plane strain problems with 3D boundary conditions for an elastic, anisotropic medium.

The derivation of the influence functions is based on the solution for a single concentrated force applied at an arbitrary point of the continuous infinite plane. This solution is the direct consequence of Lekhnitskii's one[11] for the infinite cylindrical cavity with the loaded surface in an anisotropic medium. If the resultant force is held constant as radius of the cavity is decreased to an infinitesimal quantity then Lekhnitskii's equations can be used to model action of the above-mentioned force. These equations are presented in Section 2.

The structure of Lekhnitskii's solution allows the extraction of closed-form, analytical expressions that can be integrated on an element. Such an analytical integration results in complex-valued functions that describe integrated action of forces or dipoles distributed on the element. Only these complex-valued functions have to be changed in general Lekhnitkii's equations. Therefore, when different elements are considered, we may call them as element influence functions. These functions for stress and displacement discontinuity elements are presented in Sections 3 and 4, respectively. The formulae for constant, linear, parabolic and k-order ($k > 2$) elements are produced as particular cases of the solutions for polynomially distributed stress or displacement discontinuities.

2. SINGLE CONCENTRATED FORCE. BASIC LEKHNITSKII'S EQUATIONS

Let us consider an infinite, homogeneous, linear elastic, anisotropic medium without body forces and an infinite straight line L in this medium. The line L is parallel to the axis Oz of the Cartesian system of co-ordinates $Oxyz$. (Figure 1(a)). The line is loaded by a 3-D force $\mathbf{f}\{f_x, f_y, f_z\}$ that, being evenly distributed along the line, does not depend on the coordinate z. Here $\mathbf{f}$ is the force acting at every part of such line L with the unit length; f_x, f_y, f_z are projections of the vector

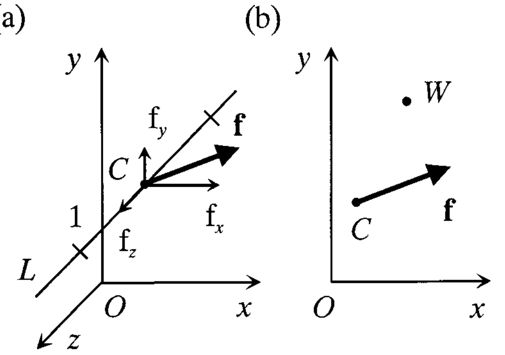


Figure 1. Calculation scheme: (a) isometric view and (b) points C and W

f on axes Ox, Oy and Oz, respectively. Elastic properties of the medium are set by the generalized Hooke's law:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} \tag{1}$$

The problem is to find the six components of the stress tensor and three components of the displacement vector at an arbitrary point in the medium due to such a loaded line.

In the above problem, induced displacement components u_x, u_y, u_z at any point of the medium are functions of x, y only; strains $\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ are, in general, non-zero; while $\varepsilon_z = 0$. There are two definitions of such a strained state of the medium. Lekhnitskii[11] called it as 'generalized plane strain'. This definition is used in Russian scientific literature. Another definition, 'complete plane strain', was introduced by Brady and Bray.[6] Although Lekhnitskii's method is employed in this work, the authors accept the last definition which is used in Western scientific literature to distinguish such a strained state.

In the problem, any plane xy, perpendicular to the axis Oz, has the same strained state. Thus, it is possible to consider the plane Oxy only. In this case, we can call the force **f** as a concentrated force applied at an arbitrary point $C(x_C, y_C)$ of the continuous infinite plane Oxy (Figure 1(b)). The infinite line L mentioned above contains the point C.

The known formulae[11–13] can be used for determining stresses and displacements at an arbitrary point $W(x_W, y_W)$ of the anisotropic plane Oxy (Figure 1(b)) due to the concentrated force $\mathbf{f}\{f_x, f_y, f_z\}$ applied at an arbitrary point $C(x_C, y_C)$:

$$\sigma_x = 2\,\mathrm{Re}[\mu_1^2\Sigma_1 + \mu_2^2\Sigma_2 + \mu_3^2\lambda_3\Sigma_3]$$

$$\sigma_y = 2\,\mathrm{Re}[\Sigma_1 + \Sigma_2 + \lambda_3\Sigma_3]$$

$$
\begin{aligned}
\sigma_z &= -(a_{13}\sigma_x + a_{23}\sigma_y + a_{34}\tau_{yz} + a_{35}\tau_{xz} + a_{36}\tau_{xy})/a_{33} \\
\tau_{yz} &= -2\,\mathrm{Re}[\lambda_1\Sigma_1 + \lambda_2\Sigma_2 + \Sigma_3] \\
\tau_{xz} &= 2\,\mathrm{Re}[\mu_1\lambda_1\Sigma_1 + \mu_2\lambda_2\Sigma_2 + \mu_3\Sigma_3] \\
\tau_{xy} &= -2\,\mathrm{Re}[\mu_1\Sigma_1 + \mu_2\Sigma_2 + \mu_3\lambda_3\Sigma_3] \\
u_x &= 2\,\mathrm{Re}\sum_{k=1}^{3} p_k\Pi_k, \qquad u_y = 2\,\mathrm{Re}\sum_{k=1}^{3} q_k\Pi_k, \qquad u_z = 2\,\mathrm{Re}\sum_{k=1}^{3} r_k\Pi_k
\end{aligned}
\tag{2}
$$

Here Re[ ] means real part of the complex number between brackets; Π_j and Σ_j are complex-valued functions

$$
\Pi_j = A_j \ln(\omega_{Wj} - \omega_{Cj}), \qquad \Sigma_j = \frac{A_j}{\omega_{Wj} - \omega_{Cj}} \tag{3}
$$

$$
\omega_{Wj} = x_W + \mu_j y_W, \qquad \omega_{Cj} = x_C + \mu_j y_C \quad (j = 1, 2, 3)
$$

Coefficients A_j are the solutions of the system of linear equations

$$
\begin{bmatrix}
1 & 1 & \lambda_3 & -1 & -1 & -\bar{\lambda}_3 \\
\mu_1 & \mu_2 & \mu_3\lambda_3 & -\bar{\mu}_1 & -\bar{\mu}_2 & -\bar{\mu}_3\bar{\lambda}_3 \\
\lambda_1 & \lambda_2 & 1 & -\bar{\lambda}_1 & -\bar{\lambda}_2 & -1 \\
p_1 & p_2 & p_3 & -\bar{p}_1 & -\bar{p}_2 & -\bar{p}_3 \\
q_1 & q_2 & q_3 & -\bar{q}_1 & -\bar{q}_2 & -\bar{q}_3 \\
r_1 & r_2 & r_3 & -\bar{r}_1 & -\bar{r}_2 & -\bar{r}_3
\end{bmatrix}
\cdot
\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \end{bmatrix}
= \frac{\mathrm{i}}{2\pi}
\begin{bmatrix} -f_y \\ f_x \\ f_z \\ 0 \\ 0 \\ 0 \end{bmatrix}
\tag{4}
$$

where i is the imaginary unit; $\pi = 3.14\ldots$; the bar denotes the complex conjugate.

In equations (2) and (4), parameters p_j, q_j, r_j are

$$
\begin{aligned}
p_j &= \beta_{11}\mu_j^2 + \beta_{12} - \beta_{16}\mu_j + \lambda_j(\beta_{15}\mu_j - \beta_{14}), \\
p_3 &= \lambda_3(\beta_{11}\mu_3^2 + \beta_{12} - \beta_{16}\mu_3) + \beta_{15}\mu_3 - \beta_{14} \\
q_j &= \beta_{12}\mu_j + \frac{\beta_{22}}{\mu_j} - \beta_{26} + \lambda_j\left(\beta_{25} - \frac{\beta_{24}}{\mu_j}\right), \\
q_3 &= \lambda_3\left(\beta_{12}\mu_3 + \frac{\beta_{22}}{\mu_3} - \beta_{26}\right) + \beta_{25} - \frac{\beta_{24}}{\mu_3} \\
r_j &= \beta_{14}\mu_j + \frac{\beta_{24}}{\mu_j} - \beta_{46} + \lambda_j\left(\beta_{45} - \frac{\beta_{44}}{\mu_j}\right), \\
r_3 &= \lambda_3\left(\beta_{14}\mu_3 + \frac{\beta_{24}}{\mu_3} - \beta_{46}\right) + \beta_{45} - \frac{\beta_{44}}{\mu_3} \quad (j = 1, 2).
\end{aligned}
\tag{5}
$$

In equations (2), (4), and (5), parameters λ_j $(j = 1, 2, 3)$ are

$$
\lambda_1 = -l_3(\mu_1)/l_2(\mu_1), \quad \lambda_2 = -l_3(\mu_2)/l_2(\mu_2), \quad \lambda_3 = -l_3(\mu_3)/l_4(\mu_3) \tag{6}
$$

where

$$
\begin{aligned}
l_2(\mu) &= \beta_{55}\mu^2 - 2\beta_{45}\mu + \beta_{44} \\
l_3(\mu) &= \beta_{15}\mu^3 - (\beta_{14} + \beta_{56})\mu^2 + (\beta_{25} + \beta_{46})\mu - \beta_{24} \\
l_4(\mu) &= \beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22}
\end{aligned} \tag{7}
$$

In equations (5) and (7), parameters β_{ij} are

$$\beta_{ij} = a_{ij} - a_{i3}a_{j3}/a_{33} \quad (i, j = 1, 2, 4, 5, 6;\ a_{ij} = a_{ji}) \tag{8}$$

Here and in equations (2), a_{ij} are rigidity coefficients in the generalized Hooke's law (1).

Complex-valued parameters μ_j ($j = 1, 2, 3$) are unequal roots with positive imaginary parts of an algebraic equation of the sixth degree:

$$l_4(\mu) \cdot l_2(\mu) - l_3^2(\mu) = b_0\mu^6 + b_1\mu^5 + b_2\mu^4 + b_3\mu^3 + b_4\mu^2 + b_5\mu + b_6 = 0 \tag{9}$$

where polynomials $l_2(\mu)$, $l_3(\mu)$, and $l_4(\mu)$ are defined by equations (7).

Thus, formulae (2) allow us to find the six components of the stress tensor and three components of the displacement vector at an arbitrary point in an infinite, homogeneous, linear elastic, anisotropic medium due to a 3-D force evenly distributed on an infinite straight line. In formulae (2), all variables with the exception of the complex-valued parameters μ_j can be found analytically from equations (3)–(8). For general anisotropic deformability of the medium, which is set by equations (1), parameters μ_j can be calculated by using known numerical methods.

If we start to construct the solution of the above problem from rigidity coefficients a_{ij} of the generalized Hooke's law (1) then, formally, we may not call Lekhnitskii's method presented in this Section as analytical. This is because we have to solve algebraic equation (9) of the sixth degree numerically in case of an arbitrary anisotropic medium. However, what is the most important, the method provides us with closed-form, analytical expressions (2) and (3) for the stress and displacement components and complex-valued functions Π_j, Σ_j once we have values a_{ij}, μ_j, p_j, q_j, r_j, λ_{ij} and β_{ij} that define elastic properties of the medium. Analytical formulae (3) allow the derivation of influence functions for stress and displacement discontinuity elements presented in Sections 3 and 4, respectively.

3. STRESS DISCONTINUITY ELEMENTS

In the paper, any element looks like an infinite flat band that is parallel to axis Oz (Figure 2(a)). Three-dimensional loads **g** or displacement discontinuities ***D*** being distributed on the band do

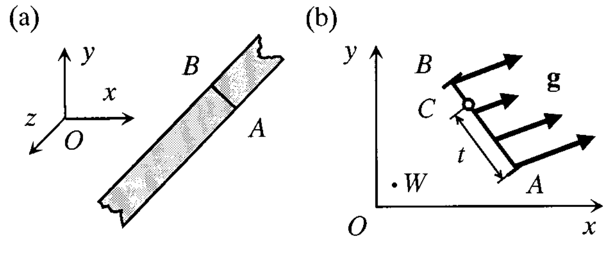


Figure 2. An isometric view on the infinite band (a) and loaded element AB on the plane Oxy (b)

not depend on the co-ordinate z. Any of the straight line segments on the band (including segment AB), which is perpendicular to its infinite side, has the same distribution of the loads or displacement discontinuities. Therefore, a complete plane-strain problem has to be considered. The straight segment AB belongs to both the band and the plane Oxy. Further in the paper, we will call the segment AB as an element, however, keeping in mind that the loads $\mathbf{g}$ or displacement discontinuities $\boldsymbol{D}$ are distributed on the infinite band.

Expressions (3) represent complex-valued functions for a 3-D single concentrated force. They have been employed to derive influence functions for stress discontinuity elements. The results have been produced by integrating force actions on the elements. During the integration, we assumed values a_{ij}, μ_j, p_j, q_j, r_j, λ_j, and β_{ij} as known and constant. Some details of the method of analytical integration and numerical examples can be found in the book of Reference 10 and the papers of References 14 and 15.

In the paper, t is the distance between point A and point C on the element, L_{AB} is the length of the segment AB and parameters ω_{Wj}, ω_{Aj}, ω_{Bj}, Λ_j, Ω_j are defined as follows:

$$\begin{aligned} &\omega_{Wj} = x_W + \mu_j y_W, \qquad \omega_{Aj} = x_A + \mu_j y_A, \qquad \omega_{Bj} = x_B + \mu_j y_B \\ &L_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\ &\Lambda_j = \frac{\omega_{Wj} - \omega_{Bj}}{\omega_{Wj} - \omega_{Aj}}, \qquad \Omega_j = \frac{\omega_{Wj} - \omega_{Aj}}{\omega_{Bj} - \omega_{Aj}}, \qquad (j = 1, 2, 3) \end{aligned} \tag{10}$$

where x_W, y_W, x_A, y_A and x_B, y_B are co-ordinates of the points W, A and B, respectively.

Presented further in this Section complex-valued functions Π_j and Σ_j are used in formulae (2) to calculate six components of the stress tensor and three components of the displacement vector at point W induced by the loaded element AB. Although the functions are valid for arbitrary positions of the element and the point under consideration ($0 \leqslant |\Omega_j| < \infty$), their Taylor series approximations for $|\Omega_j| > 1$ are also provided. The last can be useful to analyse stresses and displacements at points distant from the elements.

3.1. Polynomially distributed stress discontinuities

The load (stress) $\mathbf{g}$ changes as a polynomial function along the segment (element) AB (Figure 2(b)) and at the point C on the segment AB,

$$\mathbf{g}(t) = \sum_{k=0}^{m} \mathbf{g}_k \left(\frac{t}{L_{AB}}\right)^k, \qquad \mathbf{g}_k \equiv \mathbf{g}_k \{g_{xk}, g_{yk}, g_{zk}\} = \text{const} \tag{11}$$

where t is the distance between point A and point C on the segment AB; constants g_{xk}, g_{yk} and g_{zk} ($k = 0, 1, 2, \ldots, m$) are projections of the vectors $\mathbf{g}_k$ on axes Ox, Oy and Oz, respectively.

Complex-valued functions Π_j and Σ_j in equations (2) are

$$\begin{aligned} \Pi_j &= \sum_{k=0}^{m} \frac{A_{jk}}{k+1} \left\{ \ln \Lambda_j + \ln \Omega_j + \ln(\omega_{Bj} - \omega_{Aj}) - \Omega_j^{k+1} \left[\ln \Lambda_j + \sum_{l=1}^{k+1} \frac{\Omega_j^{-l}}{l} \right] \right\} \\ \Sigma_j &= \frac{1}{\omega_{Bj} - \omega_{Aj}} \sum_{k=0}^{m} A_{jk} \left[\frac{1}{k+1} \cdot \frac{1}{\Omega_j} - \Omega_j^k \left(\ln \Lambda_j + \sum_{l=1}^{k+1} \frac{\Omega_j^{-l}}{l} \right) \right] \end{aligned} \tag{12}$$

Coefficients $A_{jk} = A_j$ ($j = 1, 2, 3$; $k = 0, 1, \ldots, m$) are defined from the solutions of the system (4), where values f_x, f_y, f_z are substituted by $f_{xk} = L_{AB} \cdot g_{xk}$, $f_{yk} = L_{AB} \cdot g_{yk}$, $f_{zk} = L_{AB} \cdot g_{zk}$, respectively ($k = 0, 1, \ldots, m$).

If $|\Omega_j| > 1$ then

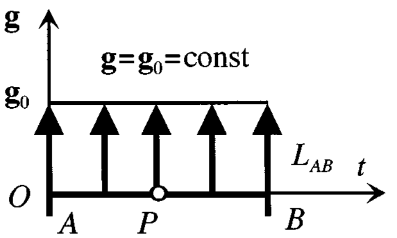
$$\ln \Lambda_j = \ln\left(1 - \frac{1}{\Omega_j}\right) = -\sum_{l=1}^{\infty} \frac{1}{l\Omega_j^l}$$

and influence functions (12) can be approximated by following Taylor series:

$$\Pi_j = \sum_{k=0}^{m} A_{jk}\left[\frac{\ln \Omega_j + \ln(\omega_{Bj} - \omega_{Aj})}{k+1} - \sum_{l=1}^{\infty} \frac{1}{l(l+1+k)} \cdot \frac{1}{\Omega_j^l}\right]$$

$$\Sigma_j = \frac{1}{\omega_{Bj} - \omega_{Aj}} \sum_{k=0}^{m} A_{jk} \sum_{l=1}^{\infty} \frac{1}{(l+k)\Omega_j^l} \tag{13}$$

3.1.1. Constant stress discontinuity element.



In the formula (11), $\mathbf{g}(t) = \mathbf{g}_0 \equiv \mathbf{g}_0\{g_{x0}, g_{y0}, g_{z0}\} = \text{const}$. Complex-valued functions Π_j and Σ_j in equations (2) are

$$\Pi_j = A_{j0}[\ln(\omega_{Bj} - \omega_{Aj}) + \ln \Omega_j + (1 - \Omega_j)\ln \Lambda_j - 1]$$

$$\Sigma_j = -\frac{A_{j0}}{\omega_{Bj} - \omega_{Aj}} \ln \Lambda_j \tag{14}$$

Coefficients $A_{j0} = A_j$ ($j = 1, 2, 3$) are defined from the solutions of the system (4), where values f_x, f_y, f_z are substituted by $f_{x0} = L_{AB} \cdot g_{x0}$, $f_{y0} = L_{AB} \cdot g_{y0}$, $f_{z0} = L_{AB} \cdot g_{z0}$, respectively.

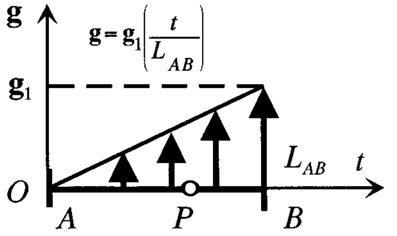
The weight centre of the element (collocation point P) is situated on the distance $t_C = \frac{1}{2} L_{AB}$ from the point A.

If $|\Omega_j| > 1$ then influence functions (14) can be approximated by following Taylor series:

$$\Pi_j = A_{j0}\left[\ln(\omega_{Bj} - \omega_{Aj}) + \ln \Omega_j + \ln \Lambda_j + \sum_{l=1}^{\infty} \frac{1}{(l+1)\Omega_j^l}\right]$$

$$\Sigma_j = -\frac{A_{j0}}{\omega_{Bj} - \omega_{Aj}} \ln \Lambda_j \tag{15}$$

3.1.2. Linear stress discontinuity element.



In the formula (11), $\mathbf{g}(t) = (t/L_{AB}) \cdot \mathbf{g}_1$, $\mathbf{g}_1 \equiv \mathbf{g}_1\{g_{x1}, g_{y1}, g_{z1}\} = \text{const}$. Complex-valued functions Π_j and Σ_j in equations (2) are

$$\Pi_j = \tfrac{1}{2} A_{j1} [\ln(\omega_{Bj} - \omega_{Aj}) + \ln \Omega_j + (1 - \Omega_j^2) \ln \Lambda_j - \Omega_j - \tfrac{1}{2}]$$

$$\Sigma_j = -\frac{A_{j1}}{\omega_{Bj} - \omega_{Aj}} (\Omega_j \ln \Lambda_j + 1) \tag{16}$$

Coefficients $A_{j1} = A_j$ $(j = 1, 2, 3)$ are defined from the solutions of the system (4), where values f_x, f_y, f_z are substituted by $f_{x1} = L_{AB} \cdot g_{x1}$, $f_{y1} = L_{AB} \cdot g_{y1}$, $f_{z1} = L_{AB} \cdot g_{z1}$, respectively.

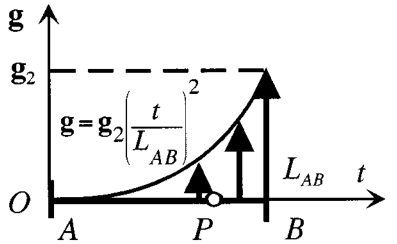
The weight centre of the element (collocation point P) is situated on the distance $t_C = \frac{2}{3} L_{AB}$ from the point A.

If $|\Omega_j| > 1$ then influence functions (16) can be approximated by following Taylor series:

$$\Pi_j = \frac{A_{j1}}{2} \left[\ln(\omega_{Bj} - \omega_{Aj}) + \ln \Omega_j + \ln \Lambda_j + \sum_{l=1}^{\infty} \frac{1}{(l+2)\Omega_j^l} \right]$$

$$\Sigma_j = -\frac{A_{j1}}{\omega_{Bj} - \omega_{Aj}} + \sum_{l=1}^{\infty} \frac{1}{(l+1)\Omega_j^l} \tag{17}$$

3.1.3. Parabolic stress discontinuity element.



In the formula (11), $\mathbf{g}(t) = (t/L_{AB})^2 \cdot \mathbf{g}_2$, $\mathbf{g}_2 \equiv \mathbf{g}_2\{g_{x2}, g_{y2}, g_{z2}\} = \text{const}$. Complex-valued functions Π_j and Σ_j in equations (2) are

$$\Pi_j = \frac{A_{j2}}{3}\left[\begin{array}{l}\ln(\omega_{Bj}-\omega_{Aj}) + \ln\Omega_j + \ln\Lambda_j - \\ -\Omega_j^3\ln\Lambda_j - \Omega_j^2 - \dfrac{\Omega_j}{2} - \dfrac{1}{3}\end{array}\right], \qquad \Sigma_j = -\frac{A_{j2}}{\omega_{Bj}-\omega_{Aj}}\left(\Omega_j^2\ln\Lambda_j + \Omega_j + \tfrac{1}{2}\right) \quad (18)$$

Coefficients $A_{j2} = A_j$ ($j = 1, 2, 3$) are defined from the solutions of the system (4), where values f_x, f_y, f_z are substituted by $f_{x2} = L_{AB} \cdot g_{x2}$, $f_{y2} = L_{AB} \cdot g_{y2}$, $f_{z2} = L_{AB} \cdot g_{z2}$, respectively.

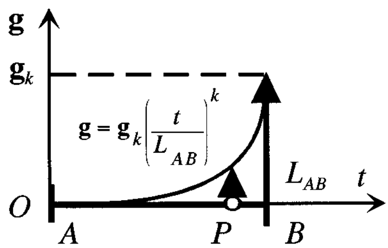
The weight centre of the element (collocation point P) is situated on the distance $t_C = \frac{3}{4} L_{AB}$ from the point A.

If $|\Omega_j| > 1$ then influence functions (18) can be approximated by the following Taylor series:

$$\Pi_j = \frac{A_{j2}}{3}\left[\ln(\omega_{Bj}-\omega_{Aj}) + \ln\Omega_j + \ln\Lambda_j + \sum_{l=1}^{\infty}\frac{1}{(l+3)\Omega_j^l}\right]$$

$$\Sigma_j = \frac{A_{j2}}{\omega_{Bj}-\omega_{Aj}} + \sum_{l=1}^{\infty}\frac{1}{(l+2)\Omega_j^l} \quad (19)$$

3.1.4. k-order (k > 2) stress discontinuity element.



In the formula (11), $\mathbf{g}(t) = (t/L_{AB})^k \cdot \mathbf{g}_k$, $\mathbf{g}_k \equiv \mathbf{g}_k\{g_{xk}, g_{yk}, g_{zk}\} = \text{const}$. ($k > 2$). Complex-valued functions Π_j and Σ_j in equations (2) are

$$\Pi_j = \frac{A_{jk}}{k+1}\left[\ln\Lambda_j + \ln\Omega_j + \ln(\omega_{Bj}-\omega_{Aj}) - \Omega_j^{k+1}\ln\Lambda_j - \sum_{l=1}^{k}\frac{\Omega_j^{k+1-l}}{l} - \frac{1}{k+1}\right] \quad (20)$$

$$\Sigma_j = -\frac{A_{jk}}{\omega_{Bj}-\omega_{Aj}}\left(\Omega_j^k\ln\Lambda_j + \sum_{l=1}^{k-1}\frac{\Omega_j^{k-l}}{l} + \frac{1}{k}\right)$$

Coefficients $A_{jk} = A_j$ ($j = 1, 2, 3$) are defined from the solutions of the system (4), where values f_x, f_y, f_z are substituted by $f_{xk} = L_{AB} \cdot g_{xk}$, $f_{yk} = L_{AB} \cdot g_{yk}$, $f_{zk} = L_{AB} \cdot g_{zk}$, respectively.

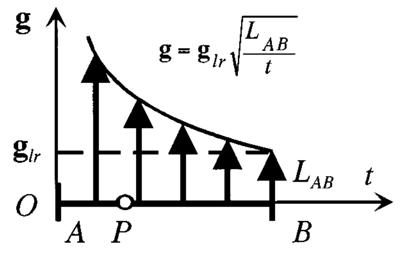
The weight centre of the element (collocation point P) is situated at a distance $t_C = [(k+1)/(k+2)]L_{AB}$ from the point A.

If $|\Omega_j| > 1$ then influence functions (20) can be approximated by the following Taylor series:

$$\Pi_j = A_{jk}\left[\frac{\ln\Omega_j + \ln(\omega_{Bj}-\omega_{Aj})}{k+1} - \sum_{l=1}^{\infty}\frac{1}{l(l+1+k)}\cdot\frac{1}{\Omega_j^l}\right]$$

$$\Sigma_j = \frac{A_{jk}}{\omega_{Bj}-\omega_{Aj}}\sum_{l=1}^{\infty}\frac{1}{(l+k)\Omega_j^l} \quad (21)$$

3.2. Reverse root stress discontinuity element



For the element,

$$\mathbf{g}(t) = \mathbf{g}_{lr}\sqrt{\frac{L_{AB}}{t}}, \qquad \mathbf{g}_{lr} \equiv \mathbf{g}_{lr}\{g_{xlr}, g_{ylr}, g_{zlr}\} = \text{const.} \tag{22}$$

Complex-valued functions Π_j and Σ_j in equations (2) are

$$\Pi_j = 2A_{jlr}\left[\ln\Lambda_j + \ln\Omega_j + \ln(\omega_{Bj} - \omega_{Aj}) - \sqrt{\Omega_j}\ln\frac{\sqrt{\Omega_j} - 1}{\sqrt{\Omega_j} + 1} - 2\right]$$
$$\Sigma_j = -\frac{A_{jlr}}{\omega_{Bj} - \omega_{Aj}}\cdot\frac{1}{\sqrt{\Omega_j}}\cdot\ln\frac{\sqrt{\Omega_j} - 1}{\sqrt{\Omega_j} + 1} \tag{23}$$

Coefficients $A_{jlr} = A_j$ ($j = 1, 2, 3$) are defined from the solutions of the system (4), where values f_x, f_y, f_z are substituted by $f_{xlr} = L_{AB}\cdot g_{xlr}$, $f_{ylr} = L_{AB}\cdot g_{ylr}$, $f_{zlr} = L_{AB}\cdot g_{zlr}$, respectively.

The weight centre of the element (collocation point P) is situated on the distance $t_C = \frac{1}{3}L_{AB}$ from the point A.

If $|\Omega_j| > 1$ then influence functions (23) can be approximated by the following Taylor series

$$\Pi_j = 2A_{jlr}\left[\ln\Lambda_j + \ln\Omega_j + \ln(\omega_{Bj} - \omega_{Aj}) + \sum_{l=1}^{\infty}\frac{2}{2l+1}\cdot\frac{1}{\Omega_j^l}\right] \tag{24}$$
$$\Sigma_j = \frac{A_{jlr}}{\omega_{Bj} - \omega_{Aj}}\cdot\sum_{l=1}^{\infty}\frac{2}{2l-1}\cdot\frac{1}{\Omega_j^l}$$

4. DISPLACEMENT DISCONTINUITY ELEMENTS

Let us consider a displacement discontinuity element (Figures 3(a) and 3(b)). The displacements $\mathbf{u}$ are continuous everywhere except over the line segment AB. Following the book[9] we can distinguish two surfaces of the element by saying that one surface is the positive one of $\eta = 0$, denoted $\eta = 0_+$, and the other is the negative one, denoted $\eta = 0_-$ (Figure 4).

The displacement discontinuity D_α ($\alpha = \xi, \eta, z$) is defined as the difference in displacement between the two surfaces of the element as follows:

$$\begin{aligned} D_\xi &= u_\xi(\xi, 0_-, z) - u_\xi(\xi, 0_+, z) \\ D_\eta &= u_\eta(\xi, 0_-, z) - u_\eta(\xi, 0_+, z) \\ D_z &= u_z(\xi, 0_-, z) - u_z(\xi, 0_+, z) \end{aligned} \tag{25}$$

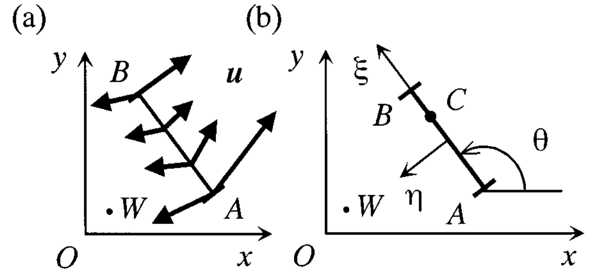


Figure 3. Displacement discontinuity element (a) and the local co-ordinate system $\xi\eta$

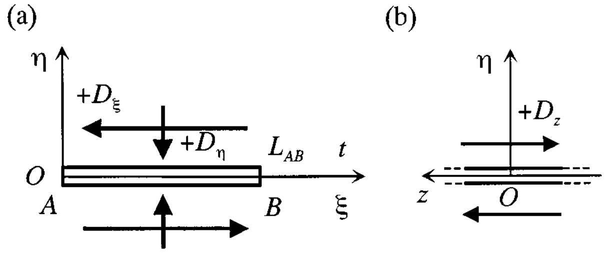


Figure 4. Positive displacement discontinuities: (a) D_ξ and D_η; (b) D_z

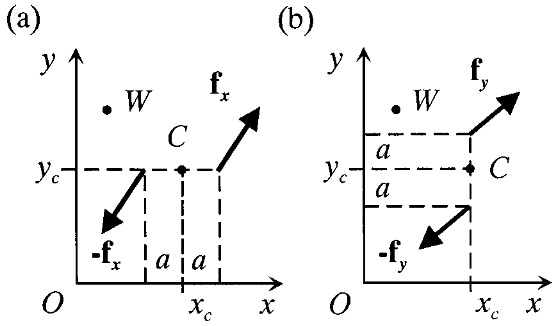


Figure 5. Stress dipoles

Because u_ξ, u_η and u_z are positive in the positive ξ, η and z co-ordinate directions, it follows that D_ξ, D_η and D_z are positive as illustrated in Figures 4(a) and 4(b), respectively.

To derive influence functions for displacement discontinuity elements, we have to integrate the point displacement discontinuity cell actions on the elements. Employing the ideas of the paper[7] let us construct these cells by using x- and y-dipoles, that are shown in Figures 5(a) and 5(b), respectively. The x-dipole is a pair of coupled concentrated forces $\mathbf{f}_x\{f_{xx}, f_{yx}, f_{zx}\}$ applied at an arbitrary point $(x_C + a, y_C)$ and $-\mathbf{f}_x\{-f_{xx}, -f_{yx}, -f_{zx}\}$ applied at the point $(x_C - a, y_C)$. The y-dipole is a pair of coupled concentrated forces $\mathbf{f}_y\{f_{xy}, f_{yy}, f_{zy}\}$ applied at the point $(x_C, y_C + a)$ and $-\mathbf{f}_y\{-f_{xy}, -f_{yy}, -f_{zy}\}$ applied at the point $(x_C, y_C - a)$. Now, we can define the displacement

discontinuity point cell applied at the point C (x_C, y_C) as a superposition of the x- and y-dipoles under conditions that the quantities $\mathbf{Q}_x = -2a\mathbf{f}_x$, $\mathbf{Q}_x \equiv \mathbf{Q}_x\{\mathrm{Q}_{xx} = -2af_{xx},\ \mathrm{Q}_{yx} = -2af_{yx},\ \mathrm{Q}_{zx} = -2af_{zx}\}$ and $\mathbf{Q}_y = -2a\mathbf{f}_y$, $\mathbf{Q}_y \equiv \mathbf{Q}_y\{\mathrm{Q}_{xy} = -2af_{xy},\ \mathrm{Q}_{yy} = -2af_{yy},\ \mathrm{Q}_{zy} = -2af_{zy}\}$ are held constant as a is decreased to an infinitesimal quantity. Let us distinguish three such cells: the first one provides displacement discontinuity $D_\xi(D_\eta = 0, D_z = 0)$ at the point C of the cell action on the element AB (Figure 3(b)); the second one provides displacement discontinuity $D_\eta(D_\xi = 0, D_z = 0)$ and the last provides displacement discontinuity $D_z(D_\xi = 0, D_\eta = 0)$.

Omitting some intermediate steps, we write down the final expressions for the complex valued functions Π_{Dj} and Σ_{Dj}. These functions, after substituting them into (2) instead of Π_j and Σ_j, allow the definition of stresses and displacements at an arbitrary point W due to the point displacement discontinuity cell actions at the point C on the element AB

$$\Pi_{Dj} = \sum_{\alpha = \xi,\eta,z} \frac{B_{\alpha j}\mu_j + C_{\alpha j}}{\omega_{Wj} - \omega_{Cj}} \cdot D_\alpha, \qquad \Sigma_{Dj} = -\sum_{\alpha = \xi,\eta,z} \frac{B_{\alpha j}\mu_j + C_{\alpha j}}{(\omega_{Wj} - \omega_{Cj})^2} \cdot D_\alpha$$

$$\omega_{Wj} = x_W + \mu_j y_W, \qquad \omega_{Cj} = x_C + \mu_j y_C \quad (\alpha = \xi, \eta, z;\ j = 1, 2, 3) \tag{26}$$

Coefficients $B_{\alpha j}$ and $C_{\alpha j}$ are the solutions of the following systems of linear equations:

$$\begin{bmatrix} 1 & 1 & \lambda_3 & -1 & -1 & -\bar{\lambda}_3 \\ \mu_1 & \mu_2 & \mu_3\lambda_3 & -\bar{\mu}_1 & -\bar{\mu}_2 & -\bar{\mu}_3\bar{\lambda}_3 \\ \lambda_1 & \lambda_2 & 1 & -\bar{\lambda}_1 & -\bar{\lambda}_2 & -1 \\ p_1 & p_2 & p_3 & -\bar{p}_1 & -\bar{p}_2 & -\bar{p}_3 \\ q_1 & q_2 & q_3 & -\bar{q}_1 & -\bar{q}_2 & -\bar{q}_3 \\ r_1 & r_2 & r_3 & -\bar{r}_1 & -\bar{r}_2 & -\bar{r}_3 \end{bmatrix} \cdot \begin{bmatrix} B_{\alpha 1} \\ B_{\alpha 2} \\ B_{\alpha 3} \\ \bar{B}_{\alpha 1} \\ \bar{B}_{\alpha 2} \\ \bar{B}_{\alpha 3} \end{bmatrix} = \frac{i}{2\pi} \begin{bmatrix} -\mathrm{Q}_{yy;\alpha} \\ \mathrm{Q}_{xy;\alpha} \\ \mathrm{Q}_{zy;\alpha} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

and

$$\begin{bmatrix} 1 & 1 & \lambda_3 & -1 & -1 & -\bar{\lambda}_3 \\ \mu_1 & \mu_2 & \mu_3\lambda_3 & -\bar{\mu}_1 & -\bar{\mu}_2 & -\bar{\mu}_3\bar{\lambda}_3 \\ \lambda_1 & \lambda_2 & 1 & -\bar{\lambda}_1 & -\bar{\lambda}_2 & -1 \\ p_1 & p_2 & p_3 & -\bar{p}_1 & -\bar{p}_2 & -\bar{p}_3 \\ q_1 & q_2 & q_3 & -\bar{q}_1 & -\bar{q}_2 & -\bar{q}_3 \\ r_1 & r_2 & r_3 & -\bar{r}_1 & -\bar{r}_2 & -\bar{r}_3 \end{bmatrix} \cdot \begin{bmatrix} C_{\alpha 1} \\ C_{\alpha 2} \\ C_{\alpha 3} \\ \bar{C}_{\alpha 1} \\ \bar{C}_{\alpha 2} \\ \bar{C}_{\alpha 3} \end{bmatrix} = \frac{i}{2\pi} \begin{bmatrix} -\mathrm{Q}_{xy;\alpha} \\ \mathrm{Q}_{xx;\alpha} \\ \mathrm{Q}_{zx;\alpha} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

respectively $(\alpha = \xi, \eta, z)$.

We have the following systems of equations to define dipole intensities:

1. Point ξ-displacement discontinuity cell. $D_\alpha = D_\xi$ $(D_\eta = 0,\ D_z = 0)$ in the expression (26).

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{14} & \beta_{15} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{24} & \beta_{25} & \beta_{26} \\ \beta_{14} & \beta_{24} & \beta_{44} & \beta_{45} & \beta_{46} \\ \beta_{15} & \beta_{25} & \beta_{45} & \beta_{55} & \beta_{56} \\ \beta_{16} & \beta_{26} & \beta_{46} & \beta_{56} & \beta_{66} \end{bmatrix} \cdot \begin{bmatrix} Q_{xx;\xi} \\ Q_{yy;\xi} \\ Q_{zy;\xi} \\ Q_{zx;\xi} \\ Q_{xy;\xi} \end{bmatrix} = \begin{bmatrix} -\sin\theta\cos\theta \\ \sin\theta\cos\theta \\ 0 \\ 0 \\ \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{29}$$

2. Point η-displacement discontinuity cell. $D_\alpha = D_\eta$ ($D_\xi = 0$, $D_z = 0$) in the expression (26):

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{14} & \beta_{15} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{24} & \beta_{25} & \beta_{26} \\ \beta_{14} & \beta_{24} & \beta_{44} & \beta_{45} & \beta_{46} \\ \beta_{15} & \beta_{25} & \beta_{45} & \beta_{55} & \beta_{56} \\ \beta_{16} & \beta_{26} & \beta_{46} & \beta_{56} & \beta_{66} \end{bmatrix} \cdot \begin{bmatrix} Q_{xx;\eta} \\ Q_{yy;\eta} \\ Q_{zy;\eta} \\ Q_{zx;\eta} \\ Q_{xy;\eta} \end{bmatrix} = \begin{bmatrix} \sin^2\theta \\ \cos^2\theta \\ 0 \\ 0 \\ -2\sin\theta\cos\theta \end{bmatrix} \tag{30}$$

3. Point z-displacement discontinuity cell. $D_\alpha = D_z$ ($D_\xi = 0$, $D_\eta = 0$) in the expression (26):

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{14} & \beta_{15} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{24} & \beta_{25} & \beta_{26} \\ \beta_{14} & \beta_{24} & \beta_{44} & \beta_{45} & \beta_{46} \\ \beta_{15} & \beta_{25} & \beta_{45} & \beta_{55} & \beta_{56} \\ \beta_{16} & \beta_{26} & \beta_{46} & \beta_{56} & \beta_{66} \end{bmatrix} \cdot \begin{bmatrix} Q_{xx;z} \\ Q_{yy;z} \\ Q_{zy;z} \\ Q_{zx;z} \\ Q_{xy;z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} \tag{31}$$

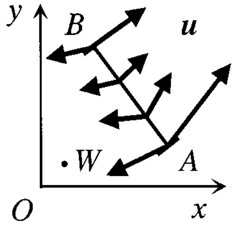
The systems of equations (29)–(31) are used to find the point stress x- and y-dipole components (here $Q_{yx;\alpha} = Q_{xy;\alpha}$, $\alpha = \xi, \eta, z$). In these systems of equations, coefficients β_{ij} ($i, j = 1, 2, 4, 5, 6$) are defined by the formulae (8) and θ is the angle between axes Ox and η (Figure 3(b))

$$\theta = \arctan\frac{y_b - y_a}{x_b - x_a} \tag{32}$$

The results presented further in this Section have been produced by integrating the displacement discontinuity point cell actions on the elements. Coefficients $B_{\alpha j}$ and $C_{\alpha j}$ ($\alpha = \xi, \eta, z$) are defined by using the solutions of the systems of equations (27)–(31), t is the distance between point A and point C on the element, L_{AB} is the length of the segment AB and the definitions (10) of the parameters $\omega_{Wj}, \omega_{Aj}, \omega_{Bj}, \Lambda_j, \Omega_j$ are also valid.

Presented further in this Section complex-valued functions Π_j and Σ_j are used in formulae (2) to calculate the six components of the stress tensor and three components of the displacement vector at point W induced by displacement discontinuities on element AB. Although the functions are valid for arbitrary positions of the element and the point under consideration ($0 \leqslant |\Omega_j| < \infty$), their Taylor series approximations for $|\Omega_j| > 1$ are provided. The last can be useful to analyse stresses and displacements at the points distant from the elements.

4.1. Polynomially distributed displacement discontinuities



The displacement discontinuity $\mathbf{D}\{D_\xi, D_\eta, D_z\}$ changes as polynomial function along the segment (element) AB and at point $C(x_C, y_C)$ on the segment AB,

$$\mathbf{D}(t) = \sum_{k=0}^{m} \mathbf{D}_k \left(\frac{t}{L_{AB}}\right)^k, \qquad \mathbf{D}_k \equiv \mathbf{D}_k\{D_{\xi k}, D_{\eta k}, D_{zk}\} = \text{const} \tag{33}$$

where t is the distance between point A and C on the element; constants $D_{\xi k}$, $D_{\eta k}$ and D_{zk} $(k = 0, 1, 2, \ldots, m)$ are displacement discontinuities in directions ξ, η and z, respectively.

Complex-valued functions Π_j and Σ_j in equations (2) are

$$\Pi_j = \frac{L_{AB}}{\omega_{Bj} - \omega_{Aj}} \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) \sum_{k=0}^{m} \Omega_j^k \left(\frac{\Omega_j^{-k-1}}{k+1} - \ln \Lambda_j - \sum_{l=1}^{k+1} \frac{\Omega_j^{-l}}{l}\right) \cdot D_{\alpha k} \tag{34}$$

$$\Sigma_j = -\frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2} \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) \sum_{k=0}^{m} \Omega_j^{k-2} \begin{bmatrix} k\Omega_j \ln \Lambda_j - \sum_{l=1}^{k+1} \left(1 - \frac{k}{l+1}\right)\Omega_j^{-l} + \\ + \frac{1}{\Lambda_j} + k - 1 + \Omega_j^{-k} \sum_{l=1}^{2} \frac{l}{k+l} \cdot \Omega_j^{1-l} \end{bmatrix} \cdot D_{\alpha k}$$

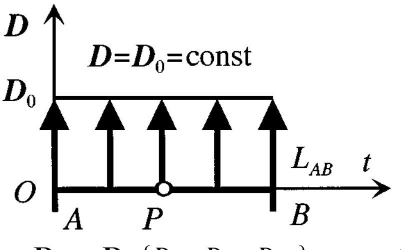
If $|\Omega_j| > 1$ then

$$\ln \Lambda_j = \ln\left(1 - \frac{1}{\Omega_j}\right) = -\sum_{l=1}^{\infty} \frac{1}{l\Omega_l^l}; \qquad \frac{1}{\Lambda_j} = \frac{1}{1 - \frac{1}{\Omega_j}} = \sum_{l=1}^{\infty} \frac{1}{\Omega_j^{l-1}}$$

and influence functions (34) can be approximated by following Taylor series:

$$\begin{aligned} \Pi_j &= \frac{L_{AB}}{\omega_{Bj} - \omega_{Aj}} \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) \sum_{k=0}^{m} \sum_{l=1}^{\infty} \frac{D_{\alpha k}}{(l+k)\Omega_j^l} \\ \Sigma_j &= -\frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2} \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) \sum_{k=0}^{m} \sum_{l=1}^{\infty} \frac{l}{l+k} \cdot \frac{D_{\alpha k}}{\Omega_j^{l+1}} \end{aligned} \tag{35}$$

4.1.1. Constant displacement discontinuity element.

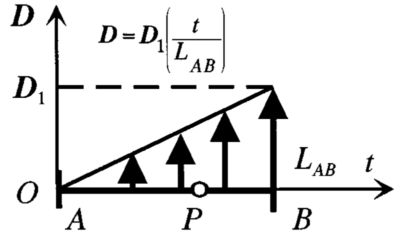


In the formula (33), $\mathbf{D}(t) = \mathbf{D}_0 \equiv \mathbf{D}_0\{D_{\xi 0}, D_{\eta 0}, D_{z0}\} = \text{const}$. Complex-valued functions Π_j and Σ_j in equations (2) are

$$\begin{aligned} \Pi_j &= -\frac{L_{AB}}{\omega_{Bj} - \omega_{Aj}} \ln \Lambda_j \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha 0} \\ \Sigma_j &= -\frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2} \cdot \frac{1}{\Lambda_j \Omega_j^2} \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha 0} \end{aligned} \tag{36}$$

The weight centre of the element (collocation point P) is situated at a distance $t_c = \frac{1}{2} L_{AB}$ from the point A.

4.1.2. Linear displacement discontinuity element.



In the formula (33), $\mathbf{D}(t) = \mathbf{D}_1 (t/L_{AB})$, $\mathbf{D}_1 \equiv \mathbf{D}_1 \{D_{\xi 1}, D_{\eta 1}, D_{z1}\} = \text{const}$. Complex-valued functions Π_j and Σ_j in equations (2) are

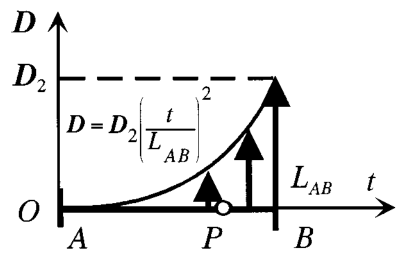
$$\begin{aligned} \Pi_j &= -\frac{L_{AB}}{\omega_{Bj} - \omega_{Aj}} (\Omega_j \ln \Lambda_j + 1) \sum_{\alpha = \xi, \eta, z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha 1} \\ \Sigma_j &= -\frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2} \left(\ln \Lambda_j + \frac{1}{\Lambda_j \Omega_j} \right) \sum_{\alpha = \xi, \eta, z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha 1} \end{aligned} \tag{37}$$

The weight centre of the element (collocation point P) is situated at a distance $t_c = \frac{2}{3} L_{AB}$ from the point A.

If $|\Omega_j| > 1$ then influence functions (37) can be approximated by following Taylor series:

$$\begin{aligned} \Pi_j &= \frac{L_{AB}}{\omega_{Bj} - \omega_{Aj}} \sum_{l=1}^{\infty} \frac{1}{(l+1)\Omega_j^l} \sum_{\alpha = \xi, \eta, z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha 1} \\ \Sigma_j &= -\frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2} \left(\ln \Lambda_j + \frac{1}{\Lambda_j \Omega_j} \right) \sum_{\alpha = \xi, \eta, z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha 1} \end{aligned} \tag{38}$$

4.1.3. Parabolic displacement discontinuity element.



In the formula (33), $\mathbf{D}(t) = \mathbf{D}_2(t/L_{AB})$, $\mathbf{D}_2 \equiv \mathbf{D}_2\{D_{\xi 2}, D_{\eta 2}, D_{z2}\} = \text{const}$. Complex-valued functions Π_j and Σ_j in equations (2) are

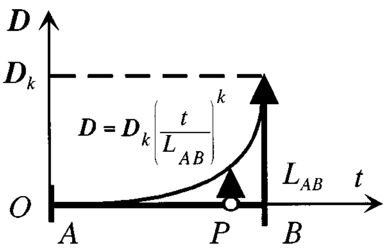
$$\Pi_j = -\frac{L_{AB}}{\omega_{Bj} - \omega_{Aj}}\left(\Omega_j^2 \ln \Lambda_j + \Omega_j + \frac{1}{2}\right) \sum_{\alpha = \xi, \eta, z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha 2}$$
$$\Sigma_j = -\frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2}\left(2\Omega_j \ln \Lambda_j + \frac{1}{\Lambda_j} + 1\right) \sum_{\alpha = \xi, \eta, z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha 2} \tag{39}$$

The weight centre of the element (collocation point P) is situated at a distance $t_c = \frac{3}{4} L_{AB}$ from the point A.

If $|\Omega_j| > 1$ then influence functions (39) can be approximated by following Taylor series:

$$\Pi_j = \frac{L_{AB}}{\omega_{Bj} - \omega_{Aj}} \cdot \frac{1}{\Omega_j}\left[\frac{1}{3} + \sum_{l=1}^{\infty} \frac{1}{(l+3)\Omega_j^l}\right] \sum_{\alpha = \xi, \eta, z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha 2}$$
$$\Sigma_j = -\frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2} \cdot \frac{1}{\Omega_j^2}\left(\frac{1}{3} + \sum_{l=1}^{\infty} \frac{l+1}{l+3} \frac{1}{\Omega_j^l}\right) \sum_{\alpha = \xi, \eta, z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha 2} \tag{40}$$

4.1.4. k-order ($k > 2$) displacement discontinuity element.



In the formula (33), $\mathbf{D}(t) = \mathbf{D}_k(t/L_{AB})^k$, $\mathbf{D}_k \equiv \mathbf{D}_k\{D_{\xi k}, D_{\eta k}, D_{zk}\} = \text{const}$. Complex-valued functions Π_j and Σ_j in equations (2) are

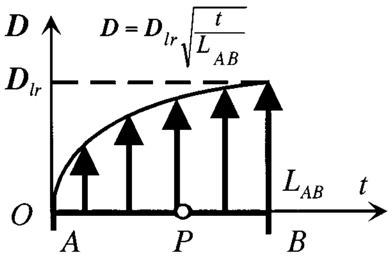
$$\Pi_j = -\frac{L_{AB}}{\omega_{Bj} - \omega_{Aj}} \cdot \left(\Omega_j^k \ln \Lambda_j + \sum_{l=1}^{k-1} \frac{\Omega_j^{k-l}}{l} + \frac{1}{k}\right) \sum_{\alpha = \xi, \eta, z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha k}$$
$$\Sigma_j = -\frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2} \cdot \begin{bmatrix} k\Omega_j^{k-1} \ln \Lambda_j + \dfrac{\Omega_j^{k-2}}{\Lambda_j} + \\ + \displaystyle\sum_{l=1}^{k-2} \frac{k-l}{l} \cdot \Omega_j^{k-1-l} + \frac{1}{k-1} \end{bmatrix} \sum_{\alpha = \xi, \eta, z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha k} \tag{41}$$

The weight centre of the element (collocation point P) is situated at a distance $t_c = (k+1)/(k+2) L_{AB}$ from the point A.

If $|\Omega_j| > 1$ then influence functions (41) can be approximated by following Taylor series:

$$\Pi_j = \frac{L_{AB}}{\omega_{Bj} - \omega_{Aj}} \sum_{l=1}^{\infty} \frac{1}{l+k} \cdot \frac{1}{\Omega_j^l} \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha k}$$
$$\Sigma_j = -\frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2} \sum_{l=1}^{\infty} \frac{l}{l+k} \cdot \frac{1}{\Omega_j^{l+1}} \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha k} \tag{42}$$

4.2. Root displacement discontinuity element



For the element,

$$\mathbf{D}(t) = \mathbf{D}_{lr}\sqrt{\frac{t}{L_{AB}}}, \qquad \mathbf{D}_{lr} \equiv \mathbf{D}_{lr}\{D_{\xi lr}, D_{\eta lr}, D_{zlr}\} = \text{const.} \tag{43}$$

Complex-valued functions Π_j and Σ_j in equations (2) are

$$\Pi_j = -\frac{2L_{AB}}{\omega_{Bj} - \omega_{Aj}} \left(1 + \frac{1}{2}\sqrt{\Omega_j}\ln\frac{\sqrt{\Omega_j} - 1}{\sqrt{\Omega_j} + 1}\right) \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha lr}$$
$$\Sigma_j = -\frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2} \left(\frac{1}{2\sqrt{\Omega_j}}\ln\frac{\sqrt{\Omega_j} - 1}{\sqrt{\Omega_j} + 1} + \frac{1}{(\sqrt{\Omega_j} + 1)(\sqrt{\Omega_j} - 1)}\right) \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha lr} \tag{44}$$

The weight centre of the element (collocation point P) is situated on the distance $t_c = \frac{3}{5} L_{AB}$ from the point A.

If $|\Omega_j| > 1$ then influence functions (44) can be approximated by the following Taylor series:

$$\Pi_j = \frac{2L_{AB}}{\omega_{Bj} - \omega_{Aj}} \sum_{l=1}^{\infty} \frac{1}{2l+1} \cdot \frac{1}{\Omega_j^l} \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha lr}$$
$$\Sigma_j = \frac{L_{AB}}{(\omega_{Bj} - \omega_{Aj})^2} \left(\sum_{l=1}^{\infty} \frac{1}{2l+1} \cdot \frac{1}{\Omega_j^l} + \frac{1}{1 - \Omega_j}\right) \sum_{\alpha=\xi,\eta,z} (B_{\alpha j}\mu_j + C_{\alpha j}) D_{\alpha lr} \tag{45}$$

5. CONCLUSIONS

Influence functions that permit us to determine stresses and displacements at an arbitrary point in an infinite, homogeneous, linear elastic, anisotropic medium due to different three-dimensional

stress or displacement discontinuities distributed on infinite, flat, band-type elements have been derived. Any straight-line segment on the band, which is perpendicular to its infinite side, had the same distribution of the discontinuities. Along with the functions, their Taylor series approximations have also been provided. The last can be useful to analyse stresses and displacements at points distant from the elements.

The new functions can be used in computer codes based on the Indirect Boundary Element Method and/or the Displacement Discontinuity Method for Geomechanics. Such codes will be able to solve complete plane strain problems with three-dimensional boundary conditions for an elastic medium with anisotropic deformability. Although complex-valued parameters μ_j (which is used to describe anisotropic deformability of the medium, see Section 2), in general, have to be calculated numerically, nevertheless, the new influence functions allow us to avoid any numerical integration procedures in the codes and, as a result, improve their accuracy.

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